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# Heavy Meson Decay Constants from Quenched Lattice QCD

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A quenched lattice QCD calculation of the $B$ and $D$ meson decay constants is presented. To investigate scaling violation associated with the heavy quarks, parallel simulations are carried out employing both Wilson and the $O(a)$-improved clover quark actions. The discretization errors due to the large $b$ quark mass are estimated with the aid of the nonrelativistic interpretation approach of El-Khadra, Kronfeld, and Mackenzie [Phys. Rev. D **55**, 3933 (1997)]. As the best values from our simulations at $\beta = 5.9$, 6.1, and 6.3 we obtain $f_B = 173(4)$ MeV, $f_{B_s} = 199(3)$ MeV for $B$ mesons and $f_D = 197(2)$ MeV, $f_{D_s} = 224(2)$ MeV for $D$ mesons where the errors are statistical. In addition we expect a 5% (7% for $D$ mesons) systematic error and a 5% error in the uncertainty to determine the lattice scale, besides the quenching error, which is not estimated in this Letter. [S0031-9007(98)06498-9]

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The $B$ meson decay constant $f_B$ is a fundamental quantity needed to extract the Cabibbo-Kobayashi-Maskawa matrix element $V_{td}$ from $B^0$-$\overline{B}^0$ mixing. For this reason lattice QCD calculations have been pursued over several years, employing either relativistic or nonrelativistic (including the static) formulation for the $b$ quark [1].

While there are a number of advantages with the relativistic formulation, its basic problem for calculations of $f_B$ lies in the difficulty to control systematic errors associated with heavy quark mass, whose magnitude in lattice units exceeds unity for the $b$ quark for a typical lattice spacing $a^{-1} \approx 2-3$ GeV accessible in current simulations. The formalism proposed in Ref. [2], however, has shed a new light on this problem: It is shown that a Wilson-type lattice quark action for heavy quark can be interpreted as a nonrelativistic Hamiltonian for an effective heavy quark field $Q$ as

$$H = \bar{Q}\left[m_1 - \frac{\vec{D}^2}{2m_2} - \frac{i\vec{\sigma}\cdot\vec{B}}{2m_B} + O(1/m_Q^2)\right]Q\,, \quad (1)$$

where the mass parameters $m_i$ $(i = 1, 2, B, \cdots)$ are functions of the bare quark mass $m_Q$ and the coupling constant. These $m_i$ are all equal in the continuum, but they mutually differ by $O(am_Q)$ at finite lattice spacing, which represents $O(am_Q)$ errors of the original action. These mass parameters are calculable in perturbation theory, and effects of $O(am_Q)$ errors on $f_B$ can be systematically analyzed. In particular, we observe that errors of $O((m_2/m_B - 1)\Lambda_{\mathrm{QCD}}/m_Q)$ for the Wilson action $(m_B \neq m_2)$ is reduced to $O(\alpha_s\Lambda_{\mathrm{QCD}}/m_Q, \Lambda_{\mathrm{QCD}}^2/m_Q^2)$ for the $O(a)$-improved clover action [1], for which $m_B = m_2$ holds at the tree level.

In this Letter we report on a calculation of the $B$ and $D$ meson decay constants in quenched lattice QCD with the relativistic formalism employing this "nonrelativistic interpretation." In order to study $O(am_Q)$ systematic errors, we carry out a parallel set of simulations using both Wilson [3] and clover quark actions. The parameters of our simulations are listed in Table I. The standard plaquette action is used to generate the gauge configurations, independently for Wilson and clover simulations. For the clover coefficient we use the tadpole-modified [4] one-loop value [5] $c_{\rm sw} = 1/u_0^3[1 + 0.199\alpha_V(1/a)]$, where $u_0 = P^{1/4}$ with $P$ the average plaquette. The lattice size is chosen to be $L \approx 2$ fm in physical units. Seven values of the heavy quark hopping parameter cover the charm and bottom quark masses, and four values for light quark in a range $0.4m_s$—$1.4m_s$ with $m_s$ strange quark mass. The simulations were carried out on the Fujitsu VPP500/80 at KEK.

The heavy-light decay constant $f_P$ is extracted from the correlators, $\langle A_4(t)P(0)\rangle$ and $\langle P(t)P(0)\rangle$, of the axial-vector current $A_4$ and the pseudoscalar density $P$. To reduce statistical errors, which rapidly increase with $am_Q$, we employ the smeared pseudoscalar density $P^S(x) = \sum_{\vec{r}} \phi(|\vec{r}|)\bar{Q}(x+r)\gamma_5 q(x)$ on the gluon configurations fixed to the Coulomb gauge. The smearing function $\phi(|\vec{r}|)$ is obtained by measuring the wave function of the pseudoscalar meson for each set of heavy and light quark masses. We are able to isolate the ground state signal from a small time separation of $t \approx 0.8$ fm. The chiral extrapolation of $f_P$ is made assuming a linear behavior against the light quark mass, which describes our data very well.

We adopt for the axial-vector current $\bar{q}\gamma_\mu\gamma_5 Q$ the one-loop renormalization factor $Z_A(am_Q)$ newly calculated with full inclusion of the heavy quark mass dependence [6]. The calculation is available for both Wilson and clover actions, and it confirms Ref. [7] made earlier for the Wilson action. The effect of finite $am_Q$ is non-negligible: with $Z_A(am_Q)$ evaluated with the coupling constant $\alpha_V(1/a)$, $f_B$ for the Wilson action is reduced by 5% ($\beta = 5.9$) to 2% ($\beta = 6.3$) compared to the calculation with the mass dependence ignored, as adopted in the previous studies. For the clover action the $am_Q$ effect works in the opposite direction with a similar magnitude.

We remark that the field $Q$ is related to the original field $\Psi$ through

$$Q = e^{am_1/2}[1 + d_1\vec{\gamma}\cdot\vec{D}]\Psi\,, \qquad (2)$$

where $d_1$ is a known function of $am_Q$ [2] and the factor $e^{am_1/2}$ includes the $m_Q$-dependent one-loop correction [6]. We ignore the $d_1\vec{\gamma}\cdot\vec{D}$ term, since its corrections to $f_B$ are at most 1%–2% due to a small $d_1(<0.1)$.

How to define the heavy meson masses is a subtle issue, since the pole mass directly measured from meson propagators suffers from large $O(am_Q)$ errors. A proposed choice is the kinetic mass $m_{\rm kin}$ defined by the energy-momentum dispersion relation of the meson,

$$E_{\rm meson}(\vec{p}) = m_{\rm pole} + \frac{\vec{p}^{\,2}}{2m_{\rm kin}} + O(\vec{p}^{\,4})\,. \qquad (3)$$

This mass, however, receives corrections from $O(\vec{p}^{\,4})$ terms in (1) which are uncontrolled and hence suffers from a large $O(am_Q)$ effect [8]. This leads to a pathology that $b$ quark mass cannot be determined consistently from heavy-light and heavy-heavy mesons [3,9].

An alternative choice is to define a "kinetic mass" by correcting the meson pole mass by the difference of the kinetic and pole masses of the heavy quark $m_2 - m_1$ [3,10],

$$m_{\rm kin} \equiv m_{\rm pole} + (m_2 - m_1)\,. \qquad (4)$$

This is motivated by the expectation that the binding energy of a heavy meson becomes independent of the heavy quark mass in the nonrelativistic limit, and $(m_2 - m_1)$ should thus represent the difference between the kinetic and the pole masses of the meson. We find that the meson mass calculated in this way does not suffer from the pathology. We adopt this definition using the one-loop calculation [6] for $m_2 - m_1$.

Let us now present our results. We plot $\Phi(m_P) = [\alpha_s(m_P)/\alpha_s(m_B)]^{2/\beta_0} f_P\sqrt{m_P}$ in Fig. 1 as a function of the inverse of the heavy meson mass $m_P$ for both Wilson (open symbols) and clover (filled symbols) actions. The light quark mass is linearly extrapolated to the chiral limit, and $\alpha_s(\mu)$ uses the standard two-loop definition where we take $\Lambda_{\rm QCD} = 295$ MeV from $\alpha_V$ from the plaquette average [4].

There is an ambiguity in practice as to what mass scale is to be adopted to represent the quantity that has mass dimension. We prefer to use a scale that facilitates a direct comparison of the $O(am_Q)$ errors with the two different quark actions for the common gauge action. Hence our natural choice is the string tension $\sigma$ for which we employ the results of Ref. [11]. Vertical lines in Fig. 1 indicate the $B$ and $D$ mesons if one uses a phenomenological value $\sqrt{\sigma} = 427$ MeV. Plotted at $1/m_P = 0$ are the static results [12], to which our data seem to converge towards the heavy quark mass limit. We observe that the Wilson results exhibit a small increase as the lattice spacing decreases, while the clover points fall almost on a single curve.

TABLE I. Simulation parameters. The lattice scale quoted is estimated from $m_\rho = 770$ MeV.

| Action | $\beta$ | 5.9 | 6.1 | 6.3 |
|---|---|---|---|---|
| | Size | $16^3 \times 40$ | $24^3 \times 64$ | $32^3 \times 80$ |
| Wilson | $N_{\rm conf}$ | 150 | 100 | 100 |
| | $1/a$ (GeV) | 2.03(3) | 2.65(4) | 3.31(6) |
| Clover | $N_{\rm conf}$ | 540 | 200 | 166 |
| | $c_{\rm sw}$ | 1.580 | 1.525 | 1.484 |
| | $1/a$ (GeV) | 1.64(2) | 2.29(4) | 3.02(5) |

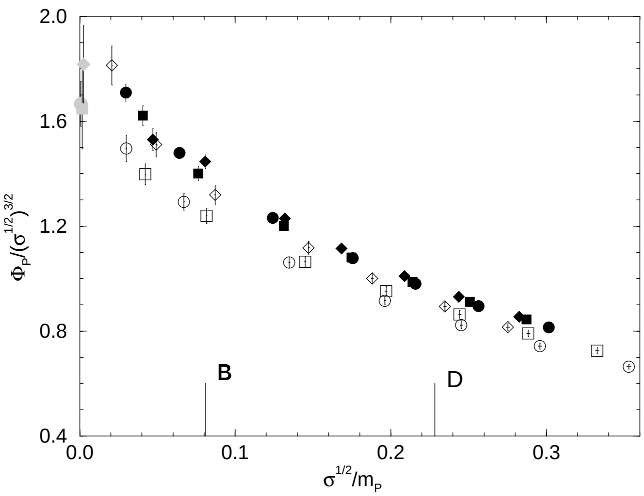


FIG. 1. $\Phi_P$ as a function of $1/m_P$ normalized by string tension $\sigma$ for Wilson (open symbols) and clover (filled symbols) action at $\beta = 5.9$ (circles), 6.1 (squares), and 6.3 (diamonds). Points at $1/m_P = 0$ show static results [12] at the same set of $\beta$.

An improved scaling behavior with the clover action is more clearly seen in Fig. 2, where we present continuum extrapolations of $f_B\sqrt{m_B}$ and $f_D\sqrt{m_D}$, which are obtained by interpolating the data in Fig. 1 to $B$ and $D$ meson masses. For the Wilson case we see scaling violation of 11%–5% for both $f_B$ and $f_D$ in our range of lattice spacing $a^{-1} \approx 1.6$–3 GeV. The clover data show a very small variation <4% over the same range. The Wilson data, when linearly extrapolated to the continuum, agree with those with the clover action within the statistical error of about 5%. We obtain $f_B\sqrt{m_B}/\sqrt{\sigma}^{3/2} = 1.399(77)$ (Wilson) and 1.406(35) (clover), and $f_D\sqrt{m_D}/\sqrt{\sigma}^{3/2} =$ 0.966(36) (Wilson) and 0.955(11) (clover) in the continuum limit, where we take the clover values being constant over the range of simulation, as no $O(a)$ scaling violation is anticipated. The error for the clover result is the dispersion of the data, and that for the Wilson result includes those associated with the extrapolation.

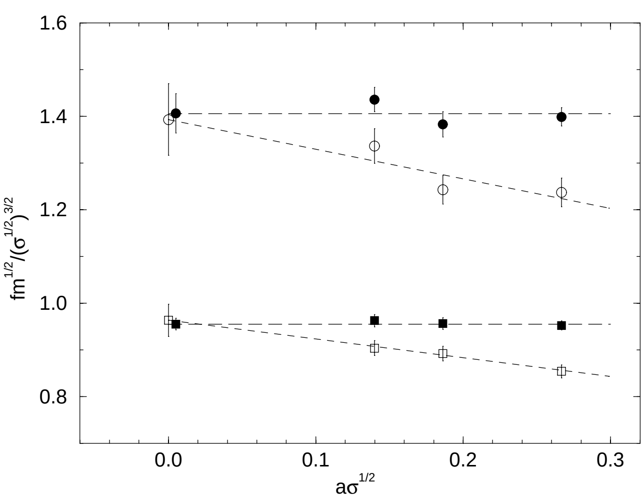


FIG. 2. Continuum extrapolation of $f_B\sqrt{m_B}$ (circles) and $f_D\sqrt{m_D}$ (squares) for Wilson (open symbols) and clover (filled symbols) action.

The linear extrapolation removes $O(a)$ errors for the Wilson action, so the remaining errors are $O(a^2)$, $O(a\alpha_s)$, and $O(\alpha_s^2)$. The last three errors also contribute to the clover results. Our present simulation does not provide a sufficient mesh in $a$ and statistical accuracies to constrain the contribution of these higher order errors. In what follows we attempt to estimate how much errors are anticipated in the clover and extrapolated Wilson results.

For the Wilson simulation, $O(a)$ scaling violation, which is removed by the extrapolation, is expected to be $O(a\Lambda_{\mathrm{QCD}}) \approx 11\%$ at our $\beta$ from a general ground. This order of magnitude is actually what we see in Fig. 2 for both $f_B$ and $f_D$. For the next orders we expect $O(\alpha_s a\Lambda_{\mathrm{QCD}})$ and $O(a^2\Lambda_{\mathrm{QCD}}^2)$, which are $O(2\%)$ and $O(1\%)$, respectively. The use of one-loop $Z_A$ leads to an additional $O(\alpha_s^2)$ uncertainty, which is $O(4\%)$ at $\beta = 6.3$. These errors altogether amount to $O(5\%)$, if added in quadrature. This estimate is admittedly crude, and the actual error could be a factor of several larger. Our data points, however, show that the deviation from a linear curve is smaller than the statistical error which is about 5%, indicating that higher order errors are not too much larger than our estimate, provided that a tricky cancellation does not take place among the error components.

The same error estimate also applies to the clover results, giving a 5% error. The data do not show a variation much beyond this error over the range of our simulation.

We must consider an error arising from the $am_Q$ effect separately, since $am_Q > 1$ and we cannot expand the effect in powers of $am_Q$. For the Wilson action this error takes the form $O((c_B - 1)\Lambda_{\mathrm{QCD}}/m_Q)$, where $c_B \equiv m_2/m_B$. The tree level value of $c_B = 1/(1 + \sinh m_1 a)$ [2] as a function of $m_2 a = e^{m_1 a}\sinh m_1 a/(1 + \sinh m_1 a)$ gives $|c_B - 1| \approx 0.7$–0.5 for $m_2 a \approx 2.9$–1.5 for the $b$ quark at $\beta = 5.9$–6.3; hence we expect an error of $O(4\%–3\%)$ in $f_B$ at our simulation points. A linear extrapolation to the continuum reduces $|c_B - 1|$ to 0.4, which indicates an $O(3\%)$ error left unremoved. For the $D$ meson, $|c_B - 1| \approx 0.4$–0.3 for the charm quark at $m_2 a \approx 0.9$–0.5 and it decreases faster, giving $|c_B - 1| \approx 0.2$ at $m_2 a = 0$. Thus, an $O(7\%–5\%)$ error for $f_D$ at our simulation points reduces to $O(3\%)$ in the continuum. We stress that the use of nonrelativistic Hamiltonian leaves an $m_Q$-dependent systematic error that cannot be removed by a linear extrapolation. We estimate it being of the order of 3% for $f_B$ and $f_D$ for the Wilson action, although we have no guarantee here that the actual error is not larger by a factor of a few.

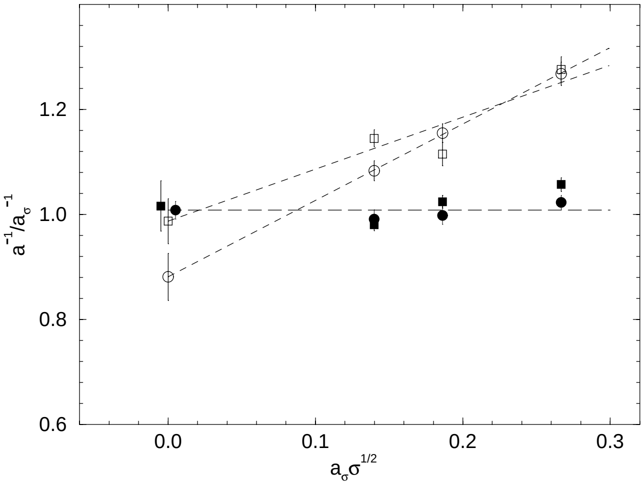


FIG. 3. Ratio of lattice scale obtained from $m_\rho$ (circles) and from $f_\pi$ (squares) to that from string tension for Wilson (open symbols) and clover (filled symbols) action.

For the clover action the $m_Q$-dependent errors are reduced to $O(\alpha_s \Lambda_{\rm QCD}/m_Q)$ and $O((\Lambda_{\rm QCD}/m_Q)^2)$. We estimate them to be $O(1\%)$ for $f_B$ and $O(4\%)$ for $f_D$. There would also be an $O(1\%-2\%)$ error from our neglect of the field rotation term (2) in the present calculation.

We now examine the question of how to set the physical scale of lattice spacing to calculate the decay constant. The most common in the literature is to use either $\rho$ meson mass $m_\rho$ or pion decay constant $f_\pi$. In Fig. 3 we give the ratio of the lattice scale obtained from $m_\rho$ or $f_\pi$ to that from the string tension. For the clover action the $O(a)$-improved axial vector current $A_4 + c_A a \partial_4 P$ is used to measure $f_\pi$ with the one-loop coefficient $c_A$ [5].

The two continuum limits of $a^{-1}/a_\sigma^{-1}$ for the Wilson action disagree by 10%, which may be ascribed to poor quality of the $f_\pi$ data. We also find a problem with the clover calculation: While we do not expect a variation proportional to $a$, we see a "gentle $a$ dependence" for this ratio. We estimate the continuum limit assuming no $O(a)$ dependence, taking the variation to be uncertainty in the scale. The error we obtain is about 3.5%, which implies 5% in the determination of $f_P\sqrt{m_P}$. (We remark that $f_\pi$ may be a quantity particularly difficult to measure, as unexpected $a$ dependences are also seen in other simulations with the clover action [13].) In spite of the problems posed here, the figure suggests that the scale error would not be larger than 10% in any case.

We present our final results for the decay constant in Table II using the scale set by $m_\rho$, and $m_s$ from kaon mass (the use of $K^*$ mass increases $m_s$, but the effect on $f_{B_s}$ or $f_{D_s}$ is only 2%, which affects little our error budget). The continuum limit is obtained by combining those of $f_P\sqrt{m_P}/\sqrt{\sigma}^{3/2}$ and $\sqrt{\sigma}/m_\rho = 0.49^{+8}_{-2}$ (Wilson) and 0.559(20) (clover), where the error for the Wilson result includes the discrepancy between the continuum value of $\sqrt{\sigma}/f_\pi$ and $\sqrt{\sigma}/m_\rho$. We remark that a direct continuum extrapolation of $f_P\sqrt{m_P}/m_\rho^{3/2}$ yields consistent results within the error ($f_\pi$ shows too large a wiggle to use for extrapolation). The errors quoted in the parentheses in Table II are, in the order given, statistical, systematic, and scale errors. The last two are indicative only, however.

TABLE II. Results for the decay constant in MeV units.

| | Wilson | Clover |
|---|---|---|
| $f_B$ | 140(11) (15) $(^{+36}_{-9})$ | 173(4) (9) (9) |
| $f_{B_s}$ | 159(10) (17) $(^{+41}_{-10})$ | 199(3) (10) (10) |
| $f_D$ | 163(13) (18) $(^{+42}_{-10})$ | 197(2) (14) (10) |
| $f_{D_s}$ | 180(11) (20) $(^{+46}_{-11})$ | 224(2) (16) (12) |

We take the result from the clover action to be our best estimate primarily because the uncertainties from scaling violation are smaller, but also our statistical sample is larger. We obtain $f_B = 173 \pm 4$ MeV and $f_{B_s} = 199 \pm 3$ MeV for the $B$ decay constants with suggested systematic uncertainty of 5% (systematic) and 5% (scale error). For the $D$ meson we obtain $f_D = 197 \pm 2$ MeV and $f_{D_s} = 224 \pm 2$ MeV with systematic and scale errors of 7% and 5%, respectively. The systematic error due to quenching is not included in our error budget.

We have shown in this Letter that heavy $B$ meson decay constant within a 10% accuracy can be obtained with the $O(a)$-improved clover quark action at $1/a \approx 1.6-3$ GeV. The systematic error associated with the heavy quark is no longer the dominant source of uncertainties. The uncertainty in the determination of the lattice scale turns out to be equally important in the quenched calculation of the heavy meson decay constant.

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